

# OPTIMAL QUANTIZERS FOR PROBABILITY DISTRIBUTIONS ON NONHOMOGENEOUS CANTOR SETS

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**ABSTRACT.** Quantization of a probability distribution refers to the idea of estimating a given probability by a discrete probability supported by a finite set. Let  $P$  be a Borel probability measure on  $\mathbb{R}$  such that  $P = \frac{1}{4}P \circ S_1^{-1} + \frac{3}{4}P \circ S_2^{-1}$ , where  $S_1$  and  $S_2$  are two similarity mappings on  $\mathbb{R}$  such that  $S_1(x) = \frac{1}{4}x$  and  $S_2(x) = \frac{1}{2}x + \frac{1}{2}$  for all  $x \in \mathbb{R}$ . Such a probability measure  $P$  has support the Cantor set generated by  $S_1$  and  $S_2$ . For this probability measure, in this paper, we give an induction formula to determine the optimal sets of  $n$ -means and the  $n$ th quantization errors for all  $n \geq 2$ . Using the induction formula we obtain some results and observations which are also given in this paper.

## 1. INTRODUCTION

Quantization of continuous random signals (or random variables and processes) is an important part of digital representation of analog signals for various coding techniques (e.g., source coding, data compression, archiving, restoration). The oldest example of quantization in statistics is rounding off. Sheppard (see [S]) was the first who analyzed rounding off for estimating densities by histograms. Any real number  $x$  can be rounded off (or quantized) to the nearest integer, say  $q(x) = [x]$ , with a resulting quantization error  $e(x) = x - q(x)$ , for example,  $q(2.14259) = 2$ . It means that the restored signal may differ from the original one and some information can be lost. Thus, in quantization of a continuous set of values there is always a distortion (also known as noise or error) between the original set of values and the quantized set of values. One of the main goals in quantization theory is to find a set of quantizers for which the distortion is minimum. For the most comprehensive overview of quantization one can see [GN] (for later references, see [GL]). Over the years several authors estimated the distortion measures for quantizers (see, e.g., [LCG] and [Z]). A class of asymptotically optimal quantizers with respect to an  $r$ th-mean error distortion measure is considered in [GL1] (see also [CG, SS1]). A different approach for uniform scalar quantization is developed in [SS2], where the correlation properties of a Gaussian process are exploited to evaluate the asymptotic behavior of the random quantization rate for uniform quantizers. General quantization problems for Gaussian processes in infinite-dimensional functional spaces are considered in [LP]. In estimating weighted integrals of time series with no quadratic mean derivatives, by means of samples at discrete times, it is known that the rate of convergence of mean-square error is reduced from  $n^{-2}$  to  $n^{-1.5}$  when the samples are quantized (see [BC1]). For smoother time series, with  $k = 1, 2, \dots$  quadratic mean derivatives, the rate of convergence is reduced from  $n^{-2k-2}$  to  $n^{-2}$  when the samples are quantized, which is a very significant reduction (see [BC2]). The interplay between sampling and quantization is also studied in [BC2], which asymptotically leads to optimal allocation between the number of samples and the number of levels of quantization. Quantization also seems to be a promising tool in recent development in numerical probability (see, e.g., [PPP]).

Let  $\mathbb{R}^d$  denote the  $d$ -dimensional Euclidean space equipped with a metric  $\|\cdot\|$  compatible with the Euclidean topology. Let  $P$  be a Borel probability measure on  $\mathbb{R}^d$  and  $\alpha$  be a finite subset of  $\mathbb{R}$ . Then,  $\int \min_{a \in \alpha} \|x - a\|^2 dP(x)$  is often referred to as the *cost*, or *distortion error* for  $\alpha$

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with respect to the probability measure  $P$ , and is denoted by  $V(P; \alpha)$ . Write  $\mathcal{D}_n := \{\alpha \subset \mathbb{R}^d : 1 \leq \text{card}(\alpha) \leq n\}$ . Then,  $\inf\{V(P; \alpha) : \alpha \in \mathcal{D}_n\}$  is called the  $n$ th quantization error for the probability measure  $P$ , and is denoted by  $V_n := V_n(P)$ . A set  $\alpha$  for which the infimum occurs and contains no more than  $n$  points is called an *optimal set of  $n$ -means*. The set of all optimal sets of  $n$ -means for a probability measure  $P$  is denoted by  $\mathcal{C}_n(P)$ . Since  $\int \|x\|^2 dP(x) < \infty$  such a set  $\alpha$  always exists (see [GKL, GL, GL1]). To know more details about quantization, one is referred to [AW, GG, GL1, GN]. For any finite  $\alpha \subset \mathbb{R}^d$ , the Voronoi region generated by an element  $a \in \alpha$  is defined as the set of all elements in  $\mathbb{R}^d$  which are closer to  $a$  than to any other element in  $\alpha$ , and is denoted by  $M(a|\alpha)$ , i.e.,

$$M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}.$$

A Borel measurable partition  $\{A_a : a \in \alpha\}$  of  $\mathbb{R}^d$  is called a *Voronoi partition* of  $\mathbb{R}^d$  with respect to  $\alpha$  (and  $P$ ) if

$$A_a \subset M(a|\alpha) \text{ (P-a.e.) for every } a \in \alpha.$$

The following proposition is known (see [GG, GL1]).

**Proposition 1.1.** *Let  $\alpha$  be an optimal set of  $n$ -means,  $a \in \alpha$ , and  $M(a|\alpha)$  be the Voronoi region generated by  $a \in \alpha$ . Then, for every  $a \in \alpha$ ,*

*(i)  $P(M(a|\alpha)) > 0$ , (ii)  $P(\partial M(a|\alpha)) = 0$ , (iii)  $a = E(X : X \in M(a|\alpha))$ , and (iv)  $P$ -almost surely the set  $\{M(a|\alpha) : a \in \alpha\}$  forms a Voronoi partition of  $\mathbb{R}^d$ .*

From the above proposition it is clear that if  $\alpha$  is an optimal set of  $n$ -means for  $P$ , and  $X$  is a random variable with the probability distribution  $P$ , then  $a$  is the expected value of the random variable  $X$  given that  $X$  is in  $M(a|\alpha)$ .

Let  $S_1, S_2 : \mathbb{R} \rightarrow \mathbb{R}$  be two contractive similarity mappings such that  $S_1(x) = r_1x$  and  $S_2(x) = r_2x + (1 - r_2)$ , where  $0 < r_1, r_2 < 1$  and  $r_1 + r_2 < 1$ . Let  $(p_1, p_2)$  be a probability vector with  $p_1, p_2 > 0$ . Then, there exists a unique Borel probability measure  $P$  on  $\mathbb{R}$  such that  $P = p_1P \circ S_1^{-1} + p_2P \circ S_2^{-1}$ , where  $P \circ S_i^{-1}$  denotes the image measure of  $P$  with respect to  $S_i$  for  $i = 1, 2$  (see [H]). For  $\sigma := \sigma_1\sigma_2 \cdots \sigma_k \in \{1, 2\}^k$ , set  $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_k}$  and  $J_\sigma := S_\sigma([0, 1])$ . Then the set  $C := \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} J_\sigma$  is known as the *Cantor set* generated by the two mappings  $S_1$  and  $S_2$ , and equals the support of the probability measure  $P$ . A Cantor set generated by two self-similar mappings is also known as a *dyadic Cantor set*. A dyadic Cantor set is called *homogeneous* if the similarity ratios of the two self-similar mappings generating the Cantor set are equal, otherwise it is called *nonhomogeneous*. If a probability distribution  $P = p_1P \circ S_1^{-1} + p_2P \circ S_2^{-1}$  has its support a homogeneous Cantor set and  $p_1 = p_2$ , then  $P$  is called a *homogeneous Cantor distribution*. The set  $C$  is known as the *classical Cantor set* if it is generated by the two self-similar mappings  $S_1(x) = \frac{1}{3}x$  and  $S_2(x) = \frac{1}{3}x + \frac{2}{3}$  for  $x \in \mathbb{R}$ . When  $r_1 = r_2 = \frac{1}{3}$  and  $p_1 = p_2 = \frac{1}{2}$ , i.e., for the self-similar probability measure  $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$  with support the classical Cantor set, Graf and Luschgy determined the optimal sets of  $n$ -means and the  $n$ th quantization error for all  $n \geq 2$  (see [GL2]). In this paper, we have taken  $r_1 = \frac{1}{4}$ ,  $r_2 = \frac{1}{2}$ ,  $p_1 = \frac{1}{4}$  and  $p_2 = \frac{3}{4}$ , i.e., the probability measure  $P$  considered here is nonhomogeneous and satisfies  $P = \frac{1}{4}P \circ S_1^{-1} + \frac{3}{4}P \circ S_2^{-1}$ , where  $S_1(x) = \frac{1}{4}x$  and  $S_2(x) = \frac{1}{2}x + \frac{1}{2}$  for  $x \in \mathbb{R}$ . For this probability measure, in this paper, we investigate the optimal sets of  $n$ -means and the  $n$ th quantization error for any positive integer  $n$ . The arrangement of the paper is as follows: Lemma 2.2, Lemma 3.1, and Lemma 3.2 give the optimal sets of  $n$ -means for  $n = 1, 2$  and 3. Proposition 3.3, Proposition 3.4, Proposition 3.7, and Proposition 3.8 give some properties about the optimal sets of  $n$ -means for all  $n \geq 2$ . Theorem 3.10 gives the induction formula to determine the optimal sets of  $n$ -means and the  $n$ th quantization error for all  $n \geq 2$ . In the last section, using the induction formula we obtain some results and observations about the optimal sets of  $n$ -means for  $n \in \mathbb{N}$ . We also gave a tree diagram of the optimal sets of  $n$ -means for a certain range of  $n$ . Finally, we would like to mention that quantization for homogeneous Cantor distributions were investigate by several authors, for example, see [GL2, K, KZ]. But, to the best

of our knowledge, the work in this paper is the first advance to investigate the quantization for nonhomogeneous Cantor distributions. The main difference with the homogeneous and the non-homogeneous systems is that for an homogeneous system there is a closed formula for optimal quantizers for  $n$ -means for all  $n \geq 2$ , but for the non-homogeneous system, considered in this paper, to obtain the optimal quantizers for  $n$ -means for all  $n \geq 2$  a closed formula is not known yet.

## 2. PRELIMINARIES

In this section, we give the basic definitions and lemmas that will be instrumental in our analysis. Let  $S_1, S_2$  be the two similarity mappings generating the Cantor set associated with the probability vector  $(p_1, p_2)$ , where  $S_1(x) = \frac{1}{4}x$  and  $S_2(x) = \frac{1}{2}x + \frac{1}{2}$  for all  $x \in \mathbb{R}$ , and  $p_1 = \frac{1}{4}$ ,  $p_2 = \frac{3}{4}$ . An *alphabet* is a finite set. By a *string* or a *word*  $\sigma$  over an alphabet  $\{1, 2\}$ , we mean a finite sequence  $\sigma := \sigma_1\sigma_2 \cdots \sigma_k$  of symbols from the alphabet, where  $k \geq 1$ , and  $k$  is called the length of the word  $\sigma$ . A word of length zero is called the *empty word*, and is denoted by  $\emptyset$ . By  $\{1, 2\}^*$  we denote the set of all words over the alphabet  $\{1, 2\}$  of some finite length  $k$  including the empty word  $\emptyset$ . By  $|\sigma|$ , we denote the length of a word  $\sigma \in \{1, 2\}^*$ . For any two words  $\sigma := \sigma_1\sigma_2 \cdots \sigma_k$  and  $\tau := \tau_1\tau_2 \cdots \tau_\ell$  in  $\{1, 2\}^*$ , by  $\sigma\tau := \sigma_1 \cdots \sigma_k\tau_1 \cdots \tau_\ell$  we mean the word obtained from the concatenation of the words  $\sigma$  and  $\tau$ . For  $\sigma, \tau \in \{1, 2\}^*$ , if  $\tau = \sigma\gamma$  for some word  $\gamma \in \{1, 2\}^*$ , then we say that  $\sigma$  is a *predecessor* of  $\tau$  and write it as  $\sigma \prec \tau$ . For  $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in \{1, 2\}^k$ ,  $k \geq 1$ , let us write

$$S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_k}, p_\sigma := p_{\sigma_1}p_{\sigma_2} \cdots p_{\sigma_k}, s_\sigma := s_{\sigma_1}s_{\sigma_2} \cdots s_{\sigma_k} \text{ and } J_\sigma := S_\sigma([0, 1]).$$

If  $\sigma$  is the empty word  $\emptyset$ , by  $S_\sigma$  we mean the identity mapping on  $\mathbb{R}$ . Then,  $P = \frac{1}{4}P \circ S_1^{-1} + \frac{3}{4}P \circ S_2^{-1}$  is a unique Borel probability measure on  $\mathbb{R}$  which has support the Cantor set  $C = \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} J_\sigma$  (see [H]). For  $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in \{1, 2\}^k$ , let us write  $c(\sigma) := \#\{i : \sigma_i = 1, 1 \leq i \leq k\}$ . Then  $\{J_\sigma\}_{\sigma \in \{1, 2\}^k}$  is the set of  $2^k$  intervals with the length of  $J_\sigma$  equals  $\lambda(J_\sigma) := \frac{1}{4^{c(\sigma)}} \frac{1}{2^{k-c(\sigma)}} = \frac{1}{2^{k+c(\sigma)}}$  at the  $k$ th level of the Cantor construction, where  $\lambda$  denotes the Lebesgue measure on  $[0, 1]$ . The intervals  $J_{\sigma_1}, J_{\sigma_2}$  into which  $J_\sigma$  is split up at the  $(k+1)$ th level are called the children of  $J_\sigma$ . Moreover, for any  $\sigma \in \{1, 2\}^*$  we have  $P(J_\sigma) = p_\sigma = \frac{3^{|\sigma|-c(\sigma)}}{4^{|\sigma|}}$ , and  $\lambda(J_\sigma) = \frac{1}{2^{|\sigma|+c(\sigma)}}$ .

Let us now prove the following lemmas.

**Lemma 2.1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Borel measurable and  $k \in \mathbb{N}$ . Then*

$$\int f dP = \sum_{\sigma \in \{1, 2\}^k} p_\sigma \int f \circ S_\sigma dP.$$

*Proof.* We know  $P = p_1P \circ S_1^{-1} + p_2P \circ S_2^{-1}$ , and so by induction  $P = \sum_{\sigma \in \{1, 2\}^k} p_\sigma P \circ S_\sigma^{-1}$ , and hence the lemma.  $\square$

**Lemma 2.2.** *Let  $X$  be a real valued random variable with distribution  $P$ . Let  $E(X)$  represent the expected value and  $V := V(X)$  represent the variance of the random variable  $X$ . Then,*

$$E(X) = \frac{2}{3} \text{ and } V(X) = \frac{16}{153}.$$

*Proof.* We have,  $E(X) = \int x dP = \frac{1}{4} \int \frac{1}{4} x dP + \frac{3}{4} \int (\frac{1}{2}x + \frac{1}{2}) dP = \frac{1}{16} E(X) + \frac{3}{8} E(X) + \frac{3}{8} = \frac{7}{16} E(X) + \frac{3}{8}$ , which implies  $E(X) = \frac{2}{3}$ . Moreover,  $E(X^2) = \int x^2 dP = \frac{1}{4} \int x^2 d(P \circ S_1^{-1}) + \frac{3}{4} \int x^2 d(P \circ S_2^{-1}) = \frac{1}{4} \int \frac{1}{16} x^2 dP + \frac{3}{4} \int (\frac{1}{2}x + \frac{1}{2})^2 dP = \frac{1}{64} E(X^2) + \frac{3}{16} E(X^2) + \frac{3}{8} E(X) + \frac{3}{16} = \frac{13}{64} E(X^2) + \frac{1}{4} + \frac{3}{16}$ , which yields  $E(X^2) = \frac{28}{51}$ , and so  $V(X) = E(X - E(X))^2 = E(X^2) - (E(X))^2 = \frac{28}{51} - (\frac{2}{3})^2 = \frac{16}{153}$ , which is the lemma.  $\square$

**Note 2.3.** For any  $x_0 \in \mathbb{R}$ , we have  $\int (x - x_0)^2 dP = V(X) + (x_0 - E(X))^2$  yielding the fact that the optimal set of one-mean is the expected value and the corresponding quantization error is the variance  $V$  of the random variable  $X$ . Since  $S_1$  and  $S_2$  are similarity mappings, we have  $E(S_j(X)) = S_j(E(X))$  for  $j = 1, 2$  and so by induction,  $E(S_\sigma(X)) = S_\sigma(E(X)) = S_\sigma(\frac{2}{3})$  for  $\sigma \in \{1, 2\}^k$ ,  $k \geq 1$ . For  $\sigma, \tau \in \{1, 2\}^*$ , write

$$a(\sigma) := E(X : X \in J_\sigma) \text{ and } a(\sigma, \tau) := E(X : X \in J_\sigma \cup J_\tau).$$

Then, using Lemma 2.1, we have

$$a(\sigma) = \frac{1}{P(J_\sigma)} \int_{J_\sigma} x dP = \sum_{\tau \in \{1, 2\}^k} \int_{J_\sigma} x d(P \circ S_\tau^{-1}) = \int_{J_\sigma} x d(P \circ S_\sigma^{-1}) = \int S_\sigma(x) dP = E(S_\sigma(X)),$$

$$\text{and similarly, } a(\sigma, \tau) = \frac{1}{P(J_\sigma \cup J_\tau)} \left( P(J_\sigma) S_\sigma\left(\frac{2}{3}\right) + P(J_\tau) S_\tau\left(\frac{2}{3}\right) \right).$$

For any  $a \in \mathbb{R}$  and  $\sigma \in \{1, 2\}^*$ , we have

$$(1) \quad \int_{J_\sigma} (x - a)^2 dP = p_\sigma \int (x - a)^2 d(P \circ S_\sigma^{-1}) = p_\sigma \left( s_\sigma^2 V + \left( S_\sigma\left(\frac{2}{3}\right) - a \right)^2 \right).$$

The equation (1) is used to determine the quantization error.

In the next section we determine the optimal sets of  $n$ -means and the  $n$ th quantization error  $V_n$  for all  $n \geq 2$ .

### 3. OPTIMAL SETS AND THE ERROR FOR ALL $n \geq 2$

In this section, we first prove some lemmas and propositions that we need to deduce Theorem 3.10 which gives the induction formula to determine the optimal sets of  $n$ -means and the  $n$ th quantization errors for all  $n \geq 2$ . To prove the lemmas and propositions, we will frequently use the equation (1).

**Lemma 3.1.** *Let  $\alpha = \{a_1, a_2\}$  be an optimal set of two-means,  $a_1 < a_2$ . Then,  $a_1 = a(1) = S_1(\frac{2}{3})$ ,  $a_2 = a(2) = S_2(\frac{2}{3})$ , and the quantization error is  $V_2 = \frac{13}{612} = 0.0212418$ .*

*Proof.* Let us first consider a two-point set  $\beta$  given by  $\beta = \{a(1), a(2)\}$ . Then,

$$\int \min_{b \in \beta} (x - b)^2 dP = \sum_{i=1}^2 \int_{J_i} (x - a(i))^2 dP = \frac{1}{64}V + \frac{3}{16}V = \frac{13}{612} = 0.0212418.$$

Since  $V_2$  is the quantization error for two-means, we have  $V_2 \leq \frac{13}{612} = 0.0212418$ . Let  $\alpha = \{a_1, a_2\}$  be an optimal set of two-means. Since  $a_1$  and  $a_2$  are the expected values of the random variable  $X$  with distribution  $P$  in their own Voronoi regions, we have  $0 \leq a_1 < a_2 \leq 1$ . Suppose that  $a_2 \leq \frac{47}{64} < \frac{3}{4} = S_{22}(0)$ . Then using (1), we have

$$\frac{13}{612} \geq V_2 = \int \min_{a \in \alpha} (x - a)^2 dP \geq \int_{J_2} (x - \frac{47}{64})^2 dP = \frac{24921}{1114112} = 0.0223685 > V_2,$$

which is a contradiction. So, we can assume that  $\frac{47}{64} < a_2$ . Since  $a_1 \geq 0$ , we have  $\frac{1}{2}(a_1 + a_2) \geq \frac{1}{2}(0 + \frac{47}{64}) = \frac{47}{128} > \frac{1}{4}$ , yielding  $a_1 \geq E(X : X \in J_1) = a(1) = \frac{1}{6}$ . If  $a_1 \geq \frac{1}{2}$ , then

$$\frac{13}{612} \geq V_2 \geq \int_{J_1} (x - \frac{1}{2})^2 dP = \frac{1}{34} = 0.0294118 > V_2,$$

which yields a contradiction. So, we can assume that  $a_1 < \frac{1}{2}$ . We now show that the Voronoi region of  $a_1$  does not contain any point from  $J_2$ . For the sake of contradiction, assume that the Voronoi region of  $a_1$  contains points from  $J_2$ , i.e.,  $\frac{1}{2}(a_1 + a_2) > \frac{1}{2}$ . Then, there exists a word  $\sigma \in \{1, 2\}^*$  such that  $S_{2\sigma 1}(1) \leq \frac{1}{2}(a_1 + a_2) \leq S_{2\sigma 2}(0)$ . Suppose that  $\sigma = 1^9 = 111111111$ , where

for any positive integer  $k$ , by  $1^k$  it is meant the word obtained from  $k$  times concatenation of the symbol 1. Thus, we have  $S_{21^9 1}(1) \leq \frac{1}{2}(a_1 + a_2) \leq S_{21^9 2}(0)$ , and so

$$a_1 = E(X : X \in J_1 \cup J_{21^9 1}) = \frac{549760532483}{3298544320512}, \text{ and}$$

$$a_2 = E(X : X \in \bigcup_{k=1}^9 J_{21^k 2} \cup J_{22}) = \frac{2621441}{3145728},$$

implying

$$V_2 = \int \min_{a \in \alpha} (x - a)^2 dP = \int_{J_1 \cup J_{21^9 1}} (x - a_1)^2 dP + \int_{\bigcup_{k=1}^9 J_{21^k 2} \cup J_{22}} (x - a_2)^2 dP$$

$$= 0.021241830065359477413 > 0.021241830065359477124 = \frac{13}{612} \geq V_2,$$

which leads to a contradiction. Similarly, we can show that a contradiction arises for any other choice of  $\sigma \in \{1, 2\}^*$ . Hence, the Voronoi region of  $a_1$  does not contain any point from  $J_2$  yielding  $a_1 \leq a(1) = \frac{1}{6}$ . Again, we have seen that  $a_1 \geq \frac{1}{6}$ . Thus, we deduce that  $a_1 = \frac{1}{6}$  and the Voronoi region of  $a_2$  does not contain any point from  $J_1$ , i.e.,  $a_2 = a(2) = \frac{5}{6}$ , and then the quantization error is  $V_2 = \frac{13}{612}$ , which is the lemma.  $\square$

**Lemma 3.2.** *Let  $\alpha = \{a_1, a_2, a_3\}$  be an optimal set of three-means such that  $a_1 < a_2 < a_3$ . Then,  $a_1 = a(1) = S_1(\frac{2}{3}) = \frac{1}{6}$ ,  $a_2 = a(21) = S_{21}(\frac{2}{3}) = \frac{7}{12}$ , and  $a_3 = a(22) = S_{22}(\frac{2}{3}) = \frac{11}{12}$ , and  $V_3 = \frac{55}{9792} = 0.00561683$ .*

*Proof.* Let us first consider a set of three points given by  $\beta := \{a(1), a(21), a(22)\}$ . The distortion error due to the set  $\beta$  is given by

$$\int \min_{a \in \beta} (x - a)^2 dP = \int_{J_1} (x - a(1))^2 dP + \int_{J_{21}} (x - a(21))^2 dP + \int_{J_{22}} (x - a(22))^2 dP = \frac{55}{9792}.$$

Since  $V_3$  is the quantization error for three-means,  $V_3 \leq \frac{55}{9792} = 0.00561683$ . Let  $\alpha := \{a_1, a_2, a_3\}$  be an optimal set of three-means. Since the optimal points are the expected values of their own Voronoi regions with respect to the probability distribution  $P$ , we have  $0 \leq a_1 < a_2 < a_3 \leq 1$ . If  $a_3 \leq \frac{27}{32} = \frac{1}{2}(S_{221}(1) + S_{222}(0))$ , then

$$V_3 \geq \int_{J_{222}} (x - \frac{27}{32})^2 dP = \frac{6939}{1114112} = 0.00622828 > V_3,$$

which leads to a contradiction. So, we can assume that  $\frac{27}{32} < a_3$ . Suppose that  $a_2 \leq \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$ . Then, for any  $x \in J_{21} = [\frac{1}{2}, \frac{1}{2} + \frac{1}{8}]$ , it follows that

$$\min_{a \in \alpha} (x - a)^2 dP = \min_{a \in \{a_2, a_3\}} (x - a)^2 \geq (x - \frac{7}{16})^2,$$

because  $x - a_2 \geq x - \frac{7}{16} \geq \frac{1}{16}$  and  $a_3 - x \geq \frac{27}{32} - x \geq \frac{7}{32} > \frac{3}{16} \geq x - \frac{7}{16} \geq \frac{1}{16}$ . Moreover,  $\frac{1}{2}(\frac{7}{16} + a(22)) = \frac{65}{96} < \frac{3}{4} = S_{22}(0)$ , and so

$$V_3 \geq \int_{J_{21}} (x - \frac{7}{16})^2 dP + \int_{J_{22}} (x - a(22))^2 dP = \frac{555}{69632} = 0.00797047 > V_3,$$

which yields a contradiction. Thus, we can assume that  $\frac{7}{16} < a_2$ . If  $a_1 \geq \frac{1}{4} + \frac{1}{16}$ , then

$$V_3 \geq \int_{J_1} (x - (\frac{1}{4} + \frac{1}{16}))^2 dP = \frac{121}{17408} = 0.00695083 > V_3,$$

which give a contradiction, and hence  $a_1 < \frac{1}{4} + \frac{1}{16}$ . Suppose that  $a_2 \geq \frac{3}{4}$ . Then, for any  $x \in J_{211}$ ,

$$\min_{a \in \{a_1, a_2\}} (x - a)^2 = (x - \frac{5}{16})^2,$$

because  $x - a_1 \geq x - \frac{5}{16}$  and  $a_2 - x \geq \frac{3}{4} - x \geq \frac{7}{32} \geq x - \frac{5}{16} \geq \frac{3}{16}$ . For any  $x \in J_{212}$ ,

$$\min_{a \in \{a_1, a_2\}} (x - a)^2 \geq (x - \frac{3}{4})^2,$$

because  $x - a_1 \geq x - \frac{5}{16} \geq \frac{1}{4} > \frac{3}{16} \geq \frac{3}{4} - x \geq \frac{1}{8}$ , and  $a_2 - x \geq \frac{3}{4} - x$ . Moreover,  $\frac{1}{2}(a(1) + a_2) \geq \frac{1}{2}(a(1) + \frac{3}{4}) = \frac{11}{24} > S_1(1) = \frac{1}{4}$ . Hence,

$$V_3 \geq \int_{J_1} (x - a(1))^2 dP + \int_{J_{211}} (x - \frac{5}{16})^2 dP + \int_{J_{212}} (x - \frac{3}{4})^2 dP = \frac{16849}{2506752} = 0.00672145 > V_3,$$

which is a contradiction. So, we can assume that  $a_2 < \frac{3}{4}$ . Suppose that  $\frac{1}{4} < a_1 < \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$ . Then, by Proposition 1.1, we have  $\frac{1}{2}(a_1 + a_2) > \frac{1}{2}$  implying  $a_2 > 1 - a_2 \geq 1 - \frac{5}{16} = \frac{11}{16} > S_{21}(1) = \frac{5}{8}$ . Hence,

$$V_3 \geq \int_{J_1} (x - \frac{1}{4})^2 dP + \int_{J_{21}} (x - \frac{11}{16})^2 dP = \frac{1193}{208896} = 0.00571098 > V_3,$$

which leads to a contradiction. So, we can assume that  $a_1 \leq \frac{1}{4}$ . Since  $\frac{1}{2}(a_1 + a_2) \geq \frac{1}{2}(0 + \frac{7}{16}) = \frac{7}{32} > \frac{13}{64} = S_{1221}(1)$ , we have  $a_1 \geq E(X \in J_{11} \cup J_{121} \cup J_{1221}) = \frac{403}{3552}$ . Now, notice that  $\frac{1}{4} < \frac{1}{2}(\frac{403}{3552} + \frac{7}{16}) < \frac{1}{2}(a_1 + a_2) < \frac{1}{2}(\frac{1}{4} + \frac{3}{4}) = \frac{1}{2}$  yielding the fact that the Voronoi region of  $a_1$  does not contain any point from  $J_2$  and the Voronoi region of  $a_2$  does not contain any point from  $J_1$ . Hence,  $a_1 = E(X : X \in J_1) = a(1) = \frac{1}{6}$ , and  $a_2 \geq E(X : X \in J_{21}) = a(21) = \frac{7}{12}$ .

Now, observe that  $S_{21}(1) = \frac{5}{8} < \frac{1}{2}(\frac{7}{12} + \frac{27}{32}) < \frac{1}{2}(a_2 + a_3) < \frac{1}{2}(\frac{3}{4} + 1) = \frac{7}{8}$  and  $\frac{7}{8} > \frac{3}{4} = S_{22}(0)$  implying the fact that the Voronoi region of  $a_2$  possibly contains points from  $J_{22}$ , but the Voronoi region of  $a_3$  does not contain any point from  $J_{21}$ . Hence,  $a_3 \geq E(X : X \in J_{22}) = a(22) = \frac{11}{12}$ . We now show that the Voronoi region of  $a_2$  does not contain any point from  $J_{22}$ . For the sake of contradiction, assume that the Voronoi region of  $a_2$  contains points from  $J_{22}$ , i.e.,  $\frac{1}{2}(a_2 + a_3) > \frac{3}{4}$ . Then, there exists a word  $\sigma \in \{1, 2\}^*$  such that  $S_{22\sigma 1}(1) \leq \frac{1}{2}(a_2 + a_3) \leq S_{22\sigma 2}(0)$ . Thus, proceeding in the similar way as in the proof of Lemma 3.1, we can show that a contradiction arises. Therefore,  $a_2$  does not contain any point from  $J_{22}$ . Hence, we have  $a_1 = a(1) = \frac{1}{6}$ ,  $a_2 = a(21) = \frac{7}{12}$ , and  $a_3 = a(22) = \frac{11}{12}$ , and the corresponding quantization error is  $V_3 = \frac{55}{9792}$ . This, completes the proof of the lemma.  $\square$

**Proposition 3.3.** *Let  $\alpha_n$  be an optimal set of  $n$ -means for  $n \geq 2$ . Then,*

$$\alpha_n \cap J_1 \neq \emptyset \text{ and } \alpha_n \cap J_2 \neq \emptyset.$$

*Moreover, the Voronoi region of any point in  $\alpha_n \cap J_1$  does not contain any point from  $J_2$ , and the Voronoi region of any point in  $\alpha_n \cap J_2$  does not contain any point from  $J_1$ .*

*Proof.* By Lemma 3.1 and Lemma 3.2, the proposition is true for  $n = 2$  and  $n = 3$ . We now prove that the proposition is true for  $n \geq 4$ . Let  $\alpha_n := \{0 \leq a_1 < a_2 < \dots < a_n \leq 1\}$  be an optimal set of  $n$ -means for  $n \geq 4$ . Consider the set of four points given by  $\beta := \{a(1), a(21), a(221), a(222)\}$ . Then,

$$\begin{aligned} \int \min_{a \in \beta} (x - a)^2 dP &= \int_{J_1} (x - a(1))^2 dP + \int_{J_{21}} (x - a(21))^2 dP + \int_{J_{221}} (x - a(221))^2 dP \\ &\quad + \int_{J_{222}} (x - a(222))^2 dP = \frac{421}{156672} = 0.00268714. \end{aligned}$$

Since  $V_n$  is the quantization error for  $n$ -means for  $n \geq 4$ , we have  $V_n \leq V_4 \leq 0.00268714$ . If  $a_1 \geq \frac{1}{4}$ , then

$$V_n \geq \int_{J_1} (x - \frac{1}{4})^2 dP = \frac{11}{3264} = 0.0033701 > V_n,$$

which leads to a contradiction. So, we can assume that  $a_1 < \frac{1}{4}$ . If  $a_n \leq \frac{1}{2}$ , then

$$V_n \geq \int_{J_2} (x - \frac{1}{2})^2 dP = \frac{7}{68} = 0.102941 > V_n,$$

which yields a contradiction, and so  $\frac{1}{2} < a_n$ . This completes the proof of the first part of the proposition. To complete the proof of the proposition, let  $j := \max\{i : a_i \leq \frac{1}{4}\}$ . Then,  $a_j \leq \frac{1}{4}$ . Suppose that the Voronoi region of  $a_j$  contains points from  $J_2$ . Then,  $\frac{1}{2}(a_j + a_{j+1}) > \frac{1}{2}$  implying  $a_{j+1} > 1 - a_j \geq 1 - \frac{1}{4} = \frac{3}{4} = S_{22}(0)$ , and so

$$V_n \geq \int_{J_{21}} \min_{a \in \alpha_n} (x-a)^2 dP = \int_{J_{21}} \min_{a \in \{a_j, a_{j+1}\}} (x-a)^2 dP \geq \int_{J_{21}} (x - \frac{3}{4})^2 dP = \frac{3}{544} = 0.00551471 > V_n,$$

which is a contradiction. Next, let  $k = \min\{i : a_i \geq \frac{1}{2}\}$  implying  $\frac{1}{2} \leq a_k$ . Assume that the Voronoi region of  $a_k$  contains points from  $J_1$ . Then,  $\frac{1}{2}(a_{k-1} + a_k) < \frac{1}{4}$  implying  $a_{k-1} < \frac{1}{2} - a_k \leq \frac{1}{2} - \frac{1}{2} = 0$ , which is a contradiction as  $0 \leq a_1 \leq a_2 < \dots < a_n \leq 1$ . Thus, the proof of the proposition is complete.  $\square$

The following proposition plays an important role in the paper.

**Proposition 3.4.** *Let  $n \geq 2$  and let  $\alpha_n$  be an optimal set of  $n$ -means such that  $\alpha_n$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$ . Set  $\alpha_1 := \alpha_n \cap J_1$ ,  $\alpha_2 := \alpha_n \cap J_2$ , and  $j := \text{card}(\alpha_1)$ . Then,  $S_1^{-1}(\alpha_1)$  is an optimal set of  $j_1$ -means and  $S_2^{-1}(\alpha_2)$  is an optimal set of  $(n - j_1)$ -means for the probability measure  $P$ , and*

$$V_n = \frac{1}{64}V_{j_1} + \frac{3}{16}V_{n-j_1}.$$

*Proof.* By Proposition 3.3, both  $\alpha_1$  and  $\alpha_2$  are nonempty sets. Moreover, since  $\alpha_n$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$ , it follows that  $\alpha_n = \alpha_1 \cup \alpha_2$ . Again, by Proposition 3.3, the Voronoi region of  $\alpha_1$  does not contain any point from  $J_2$  and the Voronoi region of  $\alpha_2$  does not contain any point from  $J_1$ . Thus, by Lemma 2.1, we have

$$\begin{aligned} V_n &= \int_{J_1} \min_{a \in \alpha_1} (x-a)^2 dP + \int_{J_2} \min_{a \in \alpha_2} (x-a)^2 dP = \frac{1}{64} \int \min_{a \in \alpha_1} (x - S_1^{-1}(a))^2 dP \\ &\quad + \frac{3}{16} \int \min_{a \in \alpha_2} (x - S_2^{-1}(a))^2 dP \\ &= \frac{1}{64} \int \min_{a \in S_1^{-1}(\alpha_1)} (x-a)^2 dP + \frac{3}{16} \int \min_{a \in S_2^{-1}(\alpha_2)} (x-a)^2 dP \end{aligned}$$

We now show that  $S_1^{-1}(\alpha_1)$  is an optimal set of  $j_1$ -means. If  $S_1^{-1}(\alpha_1)$  is not an optimal set of  $j_1$ -means, then we can find a set  $\beta \subset \mathbb{R}$  with  $\text{card}(\beta) = j_1$  such that  $\int \min_{b \in \beta} (x-b)^2 dP < \int \min_{a \in S_1^{-1}(\alpha_1)} (x-a)^2 dP$ . But, then  $S_1(\beta) \cup (\alpha_n \setminus \alpha_1)$  is a set of cardinality  $n$  such that

$$\int \min_{a \in S_1(\beta) \cup (\alpha_n \setminus \alpha_1)} (x-a)^2 dP < \int \min_{a \in \alpha_n} (x-a)^2 dP,$$

which contradicts the optimality of  $\alpha_n$ . Thus,  $S_1^{-1}(\alpha_1)$  is an optimal set of  $j_1$ -means. Similarly, one can show that  $S_2^{-1}(\alpha_2)$  is an optimal set of  $(n - j_1)$ -means. Thus, we have

$$V_n = \frac{1}{64}V_{j_1} + \frac{3}{16}V_{n-j_1},$$

which completes the proof of the proposition.  $\square$

The following lemma is needed to prove Lemma 3.6.

**Lemma 3.5.** *Let  $V(P, J_1, \{a, b\})$  be the quantization error due to the points  $a$  and  $b$  on the set  $J_1$ , where  $0 \leq a < b$  and  $b = \frac{1}{4}$ . Then,  $a = a(11, 121)$ , and*

$$V(P, J_1, \{a, b\}) = \int_{J_{11} \cup J_{121}} (x - a(11, 121))^2 dP + \int_{J_{122}} (x - \frac{1}{4})^2 dP = \frac{7711}{17547264}.$$

*Proof.* Consider the set  $\{\frac{29}{336}, \frac{1}{4}\}$ . Then, as  $S_{121}(1) < \frac{1}{2}(\frac{29}{336} + \frac{1}{4}) < S_{122}(0)$ , and  $V(P, J_1, \{a, b\})$  being the quantization error due to the points  $a$  and  $b$  on the set  $J_1$ , we have

$$V(P, J_1, \{a, b\}) \geq \int_{J_{11} \cup J_{121}} (x - \frac{29}{336})^2 dP + \int_{J_{122}} (x - \frac{1}{4})^2 dP = \frac{7711}{17547264} = 0.000439442.$$

If  $\frac{1}{8} = S_{12}(0) \leq a$ , then

$$V(P, J_1, \{a, b\}) \geq \int_{J_{11}} (x - \frac{1}{8})^2 dP = \frac{1}{2176} = 0.000459559 > V(P, J_1, \{a, b\}),$$

which is a contradiction, and so we can assume that  $a < S_{12}(0) = \frac{1}{8}$ . If the Voronoi region of  $b$  contains from  $J_{121}$ , we must have  $\frac{1}{2}(a + b) < \frac{5}{32} = S_{121}(1)$  implying  $a < \frac{5}{16} - b = \frac{5}{16} - \frac{1}{4} = \frac{1}{16}$ , and so

$$V(P, J_1, \{a, b\}) \geq \int_{J_{121}} (x - \frac{1}{16})^2 dP + \int_{J_{122}} (x - \frac{1}{4})^2 dP = \frac{125}{278528} = 0.000448788 > V(P, J_1, \{a, b\}),$$

which leads to a contradiction. So, we can assume that the Voronoi region of  $b$  does not contain any point from  $J_{121}$  yielding  $a \geq a(11, 121)$ . If the Voronoi region of  $a$  contains points from  $J_{122}$ , we must have  $\frac{1}{2}(a + \frac{1}{4}) > S_{122}(0) = \frac{3}{16}$  implying  $a > \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$ , which is a contradiction as we have seen  $a < \frac{1}{8}$ . So, the Voronoi region of  $a$  does not contain any point from  $J_{122}$  yielding  $a \leq a(11, 121)$ . Previously, we have proved  $a \geq a(11, 121)$ . Thus,  $a = a(11, 121)$  and

$$V(P, J_1, \{a, b\}) = \int_{J_{11} \cup J_{121}} (x - a(11, 121))^2 dP + \int_{J_{122}} (x - \frac{1}{4})^2 dP = \frac{7711}{17547264},$$

which is the lemma.  $\square$

**Lemma 3.6.** *Let  $\alpha$  be an optimal set of four-means. Then,  $\alpha$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$ .*

*Proof.* Let  $\alpha = \{0 \leq a_1 < a_2 < a_3 < a_4 \leq 1\}$  be an optimal set of four-means. Then, as shown in the proof of Proposition 3.3, we have  $V_4 \leq 0.00268714$ . If  $a_4 \leq S_{222}(0) = \frac{7}{8}$ , then

$$V_4 \geq \int_{J_{222}} (x - \frac{7}{8})^2 dP = \frac{63}{17408} = 0.00361903 > V_4,$$

which is a contradiction, and so  $\frac{7}{8} < a_4$ . If  $a_3 \leq \frac{1}{2}$ , then  $\frac{1}{2}(a_3 + a_4) < \frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4} = S_{22}(0)$  yielding

$$V_4 \geq \int_{J_{22}} (x - a(22))^2 dP = \frac{1}{272} = 0.00367647,$$

which is a contradiction. So, we can assume that  $a_3 > \frac{1}{2}$ . If  $a_2 \leq \frac{1}{4}$ , then,

$$V_4 \geq \int_{J_{21}} (x - a(21))^2 dP + \int_{J_{22}} (x - a(22))^2 dP = \frac{13}{3264} = 0.00398284 > V_4,$$

which leads to a contradiction, and so we have  $a_2 > \frac{1}{4}$ . If  $a_2 > \frac{1}{2}$ , there is nothing to prove because then the lemma is obviously true. First, assume that  $\frac{1}{4} < a_2 \leq \frac{3}{8}$ . Then,  $\frac{1}{2}(a_2 + a_3) > \frac{1}{2}$  yielding  $a_3 > 1 - a_2 \geq 1 - \frac{3}{8} = \frac{5}{8}$ . The following three cases can arise:

Case 1.  $\frac{5}{8} < a_3 \leq \frac{11}{16}$ .

In this case the following three subcases can arise:

Subcase (i).  $\frac{7}{8} < a_4 \leq S_{2221}(1) = \frac{29}{32}$ .

Then, as  $\frac{1}{2}(\frac{11}{16} + \frac{7}{8}) = \frac{25}{32} = S_{2212}(0)$ , we have

$$\begin{aligned} V_4 &\geq \int_{J_{21}} (x - \frac{5}{8})^2 dP + \int_{J_{2211}} (x - \frac{11}{16})^2 dP + \int_{J_{2212}} (x - \frac{7}{8})^2 dP + \int_{J_{2222}} (x - \frac{29}{32})^2 dP \\ &= \frac{14273}{4456448} = 0.00320277 > V_4, \end{aligned}$$



which leads to a contradiction.

Subcase (ii).  $S_{2221}(0) = \frac{29}{32} \leq a_4 < \frac{15}{16} = S_{2222}(0)$ .

Then, as  $\frac{1}{2}(\frac{11}{16} + \frac{29}{32}) = \frac{51}{64} = S_{22122}(0)$ , using Lemma 3.5, we have

$$\begin{aligned} V_4 &\geq \frac{7711}{17547264} + \int_{J_{21}} (x - \frac{5}{8})^2 dP + \int_{J_{2211} \cup J_{22121}} (x - \frac{11}{16})^2 dP + \int_{J_{22122} \cup J_{2221}} (x - \frac{29}{32})^2 dP \\ &\quad + \int_{J_{2222}} (x - \frac{15}{16})^2 dP = \frac{1681703}{561512448} = 0.00299495 > V_4, \end{aligned}$$

which gives a contradiction.

Subcase (iii).  $\frac{15}{16} = S_{2222}(0) \leq a_4$ .

Then, as  $\frac{1}{2}(\frac{11}{16} + \frac{15}{16}) = \frac{13}{16} = S_{221}(1)$ , using Lemma 3.5, we have

$$\begin{aligned} V_4 &\geq \frac{7711}{17547264} + \int_{J_{21}} (x - \frac{5}{8})^2 dP + \int_{J_{221}} (x - \frac{11}{16})^2 dP + \int_{J_{2221}} (x - \frac{15}{16})^2 dP \\ &= \frac{781}{274176} = 0.00284854 > V_4, \end{aligned}$$

which gives a contradiction.

Case 2.  $\frac{11}{16} < a_3 \leq \frac{3}{4}$ .

As  $a_2 \leq \frac{3}{8}$ ,  $\frac{7}{8} \leq a_4$  and  $\frac{1}{2}(\frac{3}{8} + \frac{11}{16}) = \frac{17}{32} = S_{211}(1)$ , and  $\frac{1}{2}(\frac{3}{4} + \frac{7}{8}) = \frac{13}{16} = S_{221}(1)$ , we have

$$V_4 \geq \int_{J_{211}} (x - \frac{3}{8})^2 dP + \int_{J_{212}} (x - \frac{11}{16})^2 dP + \int_{J_{221}} (x - \frac{3}{4})^2 dP + \int_{J_{222}} (x - a(222))^2 dP = \frac{843}{278528},$$

i.e.,  $V_4 \geq 0.00302663 > V_4$  which is a contradiction.

Case 3.  $\frac{3}{4} < a_3$ .

Then, as  $\frac{1}{2}(\frac{3}{4} + \frac{7}{8}) = \frac{13}{16} = S_{221}(1)$ , the Voronoi region of  $a_4$  does not contain any point from  $J_{221}$ . Again  $\frac{1}{2}(a_2 + a_3) \leq \frac{1}{2}(\frac{3}{8} + \frac{3}{4}) = \frac{9}{16} = S_{212}(0)$ . This yields the fact that

$$V_4 \geq \int_{J_{212}} (x - \frac{3}{4})^2 dP + \int_{J_{221}} (x - a(221))^2 dP = \frac{865}{278528} = 0.00310561 > V_4,$$

which yields a contradiction. Thus, the assumption  $\frac{1}{4} < a_2 \leq \frac{3}{8}$  is not correct. Suppose that  $\frac{3}{8} \leq a_2 < \frac{1}{2}$ . Then,  $\frac{1}{2}(a_1 + a_2) < \frac{1}{4}$  implying  $a_1 < \frac{1}{2} - a_2 \leq \frac{1}{2} - \frac{3}{8} = \frac{1}{8} = S_{12}(0)$ . Suppose that  $a_3 \leq \frac{11}{16}$ . Then, as  $\frac{7}{8} < a_4$ , we have  $\frac{1}{2}(\frac{11}{16} + \frac{7}{8}) = \frac{25}{32} = S_{2212}(0)$  yielding the fact that

$$V_4 \geq \int_{J_{12}} (x - \frac{1}{8})^2 dP + \int_{J_{2211}} (x - \frac{11}{16})^2 dP + \int_{J_{2212}} (x - \frac{7}{8})^2 dP + \int_{J_{222}} (x - a(222))^2 dP = \frac{1703}{557056},$$

implying  $V_4 \geq 0.00305714 > V_4$ , which is a contradiction. Now, suppose that  $\frac{11}{16} < a_3 \leq \frac{3}{4}$ . Then, as  $\frac{1}{2}(\frac{1}{2} + \frac{11}{16}) = \frac{19}{32} = S_{2122}(0)$  and  $S_{221}(1) = \frac{13}{16} = \frac{1}{2}(\frac{3}{4} + \frac{7}{8})$ , we have

$$V_4 \geq \int_{J_{12}} (x - \frac{1}{8})^2 dP + \int_{J_{211} \cup J_{2121}} (x - \frac{1}{2})^2 dP + \int_{J_{2122}} (x - \frac{11}{16})^2 dP + \int_{J_{221}} (x - \frac{3}{4})^2 dP = \frac{1501}{557056},$$

which implies  $V_4 \geq \frac{1501}{557056} = 0.00269452 > V_4$  yielding a contradiction. Next, assume that  $\frac{3}{4} < a_3$ . Then, as the Voronoi region of  $a_4$  does not contain any point from  $J_{221}$  and  $\frac{1}{2}(\frac{1}{2} + \frac{3}{4}) = \frac{5}{8}$ , we have

$$V_4 \geq \int_{J_{12}} (x - \frac{1}{8})^2 dP + \int_{J_{21}} (x - \frac{1}{2})^2 dP + \int_{J_{221}} (x - a(221))^2 dP = \frac{57}{17408} = 0.00327436 > V_4,$$

which gives a contradiction. Thus, the assumption  $\frac{3}{8} \leq a_2 < \frac{1}{2}$  leads to a contradiction. Hence, we can assume that  $\frac{1}{2} \leq a_2$ . Again, we have seen  $a_1 \leq \frac{1}{4}$ . Thus, we see that  $\alpha$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$ , which is the lemma.  $\square$

**Proposition 3.7.** *Let  $\alpha_n$  be an optimal set of  $n$ -means for any  $n \geq 2$ . Then,  $\alpha_n$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$ .*

*Proof.* By Lemma 3.1, Lemma 3.2 and Lemma 3.6, the proposition is true for all  $2 \leq n \leq 4$ . Proceeding in the similar way as Lemma 3.6, we can show that the proposition is also true for  $n = 5$  and  $n = 6$ . Now, we prove that the proposition is true for all  $n \geq 7$ . Let  $\alpha_n$  be an optimal set of  $n$ -means for any  $n \geq 7$ . Let us consider the set of seven points  $\beta := \{a(11), a(12), a(211), a(212), a(221), a(2221), a(2222)\}$ . The distortion error due to the set  $\beta$  is

$$\begin{aligned} & \int \min_{a \in \beta} (x - a)^2 dP \\ &= \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{12}} (x - a(12))^2 dP + \int_{J_{211}} (x - a(211))^2 dP + \int_{J_{212}} (x - a(212))^2 dP \\ &+ \int_{J_{221}} (x - a(221))^2 dP + \int_{J_{2221}} (x - a(2221))^2 dP + \int_{J_{2222}} (x - a(2222))^2 dP = \frac{1483}{2506752}. \end{aligned}$$

Since  $V_7$  is the quantization error for seven-means, we have  $V_7 \leq \frac{1483}{2506752} = 0.000591602$ . Then, for  $n \geq 7$ , we have  $V_n \leq V_7 \leq 0.000591602$ . Let  $j = \max\{i : a_i < \frac{1}{2} \text{ for all } 1 \leq i \leq n\}$ . Then,  $a_j < \frac{1}{2}$ . Since  $\alpha_n \cap J_1 \neq \emptyset$  and  $\alpha_n \cap J_2 \neq \emptyset$ , we have  $2 \leq j \leq n - 1$ . We need to show that  $a_j \leq \frac{1}{4}$ . Suppose that  $a_j \in (\frac{1}{4}, \frac{1}{2})$ . Then, two cases can arise:

Case 1.  $\frac{1}{4} < a_j \leq \frac{3}{8}$ .

Then,  $\frac{1}{2}(a_j + a_{j+1}) > \frac{1}{2}$  implying  $a_{j+1} > 1 - a_j \geq 1 - \frac{3}{8} = \frac{5}{8} = S_{21}(1)$ , and so

$$V_n \geq \int_{J_{21}} (x - \frac{5}{8})^2 dP = \frac{11}{17408} = 0.000631893 > V_n,$$

which is a contradiction.

Case 2.  $\frac{3}{8} \leq a_j < \frac{1}{2}$ .

Then,  $\frac{1}{2}(a_{j-1} + a_j) < \frac{1}{4}$  implying  $a_{j-1} < \frac{1}{2} - a_j \leq \frac{1}{2} - \frac{3}{8} = \frac{1}{8} = S_{12}(0)$ , and so

$$V_n \geq \int_{J_{12}} (x - \frac{1}{8})^2 dP = \frac{7}{4352} = 0.00160846 > V_n,$$

which yields a contradiction.

By Case 1 and Case 2, we can assume that  $\alpha_n$  does not contain any point from the open interval  $(\frac{1}{4}, \frac{1}{2})$  which completes the proof of the proposition.  $\square$

**Proposition 3.8.** *Let  $\alpha_n$  be an optimal set of  $n$ -means for  $n \geq 2$ . Then, for  $c \in \alpha_n$ , we have  $c = a(\tau)$  for some  $\tau \in \{1, 2\}^*$ .*

*Proof.* Let  $\alpha_n$  be an optimal set of  $n$ -means for  $n \geq 2$  and  $c \in \alpha_n$ . Then, by Proposition 3.4, we see that either  $c \in \alpha_n \cap J_1$  or  $c \in \alpha_n \cap J_2$ . Without any loss of generality, we can assume that  $c \in \alpha_n \cap J_1$ . If  $\text{card}(\alpha_n \cap J_1) = 1$ , then by Proposition 3.4,  $S_1^{-1}(\alpha_n \cap J_1)$  is an optimal set of one-mean yielding  $c = S_1(\frac{2}{3}) = a(1)$ . Assume that  $\text{card}(\alpha_n \cap J_1) \geq 2$ . Then, as similarity mappings preserve the ratio of the distances of a point from any other two points, using Proposition 3.4 again, we have  $(\alpha_n \cap J_1) \cap J_{11} = \alpha_n \cap J_{11} \neq \emptyset$  and  $(\alpha_n \cap J_1) \cap J_{12} = \alpha_n \cap J_{12} \neq \emptyset$ , and  $\alpha_n \cap J_1$  does not contain any point from the open interval  $(S_{11}(1), S_{12}(0))$  yielding the fact that  $c \in (\alpha_n \cap J_{11}) \cup (\alpha_n \cap J_{12})$ . Without any loss of generality, assume that  $c \in \alpha_n \cap J_{11}$ . If  $\text{card}(\alpha_n \cap J_{11}) = 1$ , by Proposition 3.4 as before, we see that  $S_{11}^{-1}(\alpha_n \cap J_{11})$  is an optimal set of one-mean implying  $c = S_{11}(\frac{2}{3})$ . If  $\text{card}(\alpha_n \cap J_{11}) \geq 2$ , then proceeding inductively in the similar way, we can find a word  $\tau \in \{1, 2\}^*$  with  $11 \prec \tau$ , such that  $c \in \alpha_n \cap J_\tau$  and  $\text{card}(\alpha_n \cap J_\tau) = 1$ , and then  $S_\tau^{-1}(\alpha_n \cap J_\tau)$  being an optimal set of one-mean for  $P$ , we have  $c = S_\tau(\frac{2}{3}) = a(\tau)$ . Thus, the proof of the proposition is yielded.  $\square$

We need the following lemma to prove the main theorem Theorem 3.10.

**Lemma 3.9.** *Let  $\sigma, \tau \in \{1, 2\}^*$ . Then*

$$\begin{aligned} & P(J_{\sigma 1})(\lambda(J_{\sigma 1}))^2 + P(J_{\sigma 2})(\lambda(J_{\sigma 2}))^2 + P(J_\tau)(\lambda(J_\tau))^2 \\ & < P(J_\sigma)(\lambda(J_\sigma))^2 + P(J_{\tau 1})(\lambda(J_{\tau 1}))^2 + P(J_{\tau 2})(\lambda(J_{\tau 2}))^2 \end{aligned}$$

if and only if  $P(J_\sigma)(\lambda(J_\sigma))^2 > P(J_\tau)(\lambda(J_\tau))^2$ .

*Proof.* For any  $\eta \in \{1, 2\}^*$ , we have  $P(J_{\eta 1}) = \frac{1}{4}P(J_\eta)$ ,  $P(J_{\eta 2}) = \frac{3}{4}P(J_\eta)$ ,  $\lambda(J_{\eta 1}) = \frac{1}{4}\lambda(J_\eta)$ , and  $\lambda(J_{\eta 2}) = \frac{1}{2}\lambda(J_\eta)$ . Then, for  $\sigma, \tau \in \{1, 2\}^*$ ,

$$\begin{aligned} & (P(J_{\sigma 1})(\lambda(J_{\sigma 1}))^2 + P(J_{\sigma 2})(\lambda(J_{\sigma 2}))^2 + P(J_\tau)(\lambda(J_\tau))^2) \\ & \quad - (P(J_\sigma)(\lambda(J_\sigma))^2 + P(J_{\tau 1})(\lambda(J_{\tau 1}))^2 + P(J_{\tau 2})(\lambda(J_{\tau 2}))^2) \\ &= \left( \frac{1}{64}P(J_\sigma)(\lambda(J_\sigma))^2 + \frac{3}{16}P(J_\sigma)(\lambda(J_\sigma))^2 + P(J_\tau)(\lambda(J_\tau))^2 \right) \\ & \quad - \left( P(J_\sigma)(\lambda(J_\sigma))^2 + \frac{1}{64}P(J_\tau)(\lambda(J_\tau))^2 + \frac{3}{16}P(J_\tau)(\lambda(J_\tau))^2 \right) \\ &= \frac{1}{64} (P(J_\sigma)(\lambda(J_\sigma))^2 - P(J_\tau)(\lambda(J_\tau))^2) + \frac{3}{16} (P(J_\sigma)(\lambda(J_\sigma))^2 - P(J_\tau)(\lambda(J_\tau))^2) \\ & \quad - (P(J_\sigma)(\lambda(J_\sigma))^2 - P(J_\tau)(\lambda(J_\tau))^2) \\ &= -\frac{51}{64} (P(J_\sigma)(\lambda(J_\sigma))^2 - P(J_\tau)(\lambda(J_\tau))^2), \end{aligned}$$

and thus, the lemma follows.  $\square$

Due to Proposition 3.8 and Lemma 3.9, we are now ready to state and prove the following theorem, which gives the induction formula to determine the optimal sets of  $n$ -means and the  $n$ th quantization error for all  $n \geq 2$ .

**Theorem 3.10.** *For any  $n \geq 2$ , let  $\alpha_n$  be an optimal set of  $n$ -means for  $P$ , i.e.,  $\alpha_n \in \mathcal{C}_n := \mathcal{C}_n(P)$ . Set  $O_n(\alpha_n) := \{\sigma \in \{1, 2\}^* : S_\sigma(\frac{2}{3}) \in \alpha_n\}$ , and*

$$\hat{O}_n(\alpha_n) := \{\tau \in O_n(\alpha_n) : P(J_\tau)(\lambda(J_\tau))^2 \geq P(J_\sigma)(\lambda(J_\sigma))^2 \text{ for all } \sigma \in O_n(\alpha_n)\}.$$

*Take any  $\tau \in \hat{O}_n(\alpha_n)$ . Then,  $\alpha_{n+1}(\tau) := \{S_\sigma(\frac{2}{3}) : \sigma \in (O_n(\alpha_n) \setminus \{\tau\})\} \cup \{S_{\tau 1}(\frac{2}{3}), S_{\tau 2}(\frac{2}{3})\}$  is an optimal set of  $(n+1)$ -means for  $P$ , and the number of such sets is given by*

$$\text{card}\left(\bigcup_{\alpha_n \in \mathcal{C}_n} \{\alpha_{n+1}(\tau) : \tau \in \hat{O}_n(\alpha_n)\}\right).$$

Moreover, the  $n$ th quantization error is given by

$$V_n = \sum_{\sigma \in O_n(\alpha_n)} P(J_\sigma)(\lambda(J_\sigma))^2 V = \sum_{\sigma \in O_n} \frac{3^{|\sigma|-c(\sigma)}}{2^{4|\sigma|+2c(\sigma)}} V.$$

*Proof.* By Lemma 3.1 and Lemma 3.2, we see that the optimal sets of two- and three-means are  $\{S_1(\frac{2}{3}), S_2(\frac{2}{3})\}$  and  $\{S_1(\frac{2}{3}), S_{21}(\frac{2}{3}), S_{22}(\frac{2}{3})\}$ . Notice that

$$P(J_2)(\lambda(J_2))^2 = \frac{3}{4} \left(\frac{1}{2}\right)^2 > \frac{1}{4} \left(\frac{1}{4}\right)^2 = P(J_1)(\lambda(J_1))^2.$$

Moreover,

$$V_2 = P(J_1)(\lambda(J_1))^2 V + P(J_2)(\lambda(J_2))^2 V = \frac{1}{4} \left(\frac{1}{4}\right)^2 V + \frac{3}{4} \left(\frac{1}{2}\right)^2 V = \sum_{\sigma \in \{1, 2\}} \frac{3^{|\sigma|-c(\sigma)}}{2^{4|\sigma|+2c(\sigma)}} V = \frac{13}{612}.$$

Hence, the theorem is true for  $n = 2$ . For any  $n \geq 2$ , let us now assume that  $\alpha_n$  is an optimal set of  $n$ -means. Set  $O_n(\alpha_n) := \{\sigma \in \{1, 2\}^* : S_\sigma(\frac{2}{3}) \in \alpha_n\}$ , and  $\hat{O}_n(\alpha_n) := \{\tau \in O_n(\alpha_n) : P(J_\tau)(\lambda(J_\tau))^2 \geq P(J_\sigma)(\lambda(J_\sigma))^2 \text{ for all } \sigma \in O_n(\alpha_n)\}$ . If  $\tau \notin \hat{O}_n(\alpha_n)$ , i.e., if  $\tau \in O_n(\alpha_n) \setminus \hat{O}_n(\alpha_n)$ , then by Lemma 3.9, the error

$$\int \min_{\sigma \in (O_n(\alpha_n) \setminus \{\tau\}) \cup \{\tau 1, \tau 2\}} (x - S_\sigma(\frac{2}{3}))^2 dP$$

obtained in this case is strictly greater than the corresponding error obtained in the case where  $\tau \in \hat{O}_n(\alpha_n)$ . Hence for any  $\tau \in \hat{O}_n(\alpha_n)$  the set  $\alpha_{n+1}(\tau) = \{S_\sigma(\frac{2}{3}) : \sigma \in (O_n(\alpha_n) \setminus \{\tau\})\} \cup \{S_{\tau_1}(\frac{2}{3}), S_{\tau_2}(\frac{2}{3})\}$  is an optimal set of  $(n+1)$ -means for  $P$ , and the number of such sets equals  $\text{card}\left(\bigcup_{\alpha_n \in \mathcal{C}_n(P)} \{\alpha_{n+1}(\tau) : \tau \in \hat{O}_n(\alpha_n)\}\right)$ . Moreover, the  $n$ th quantization error is given by

$$\begin{aligned} V_n &= \int \min_{\sigma \in O_n(\alpha_n)} \left(x - S_\sigma\left(\frac{2}{3}\right)\right)^2 dP = \sum_{\sigma \in O_n(\alpha_n)} \int_{J_\sigma} \left(x - S_\sigma\left(\frac{2}{3}\right)\right)^2 dP \\ &= \sum_{\sigma \in O_n(\alpha_n)} P(J_\sigma) (\lambda(J_\sigma))^2 V = \sum_{\sigma \in O_n(\alpha_n)} \frac{3^{|\sigma|-c(\sigma)}}{4^{|\sigma|}} \left(\frac{1}{2^{|\sigma|+c(\sigma)}}\right)^2 V = \sum_{\sigma \in O_n(\alpha_n)} \frac{3^{|\sigma|-c(\sigma)}}{2^{4|\sigma|+2c(\sigma)}} V. \end{aligned}$$

Thus, the proof of the theorem is complete.  $\square$

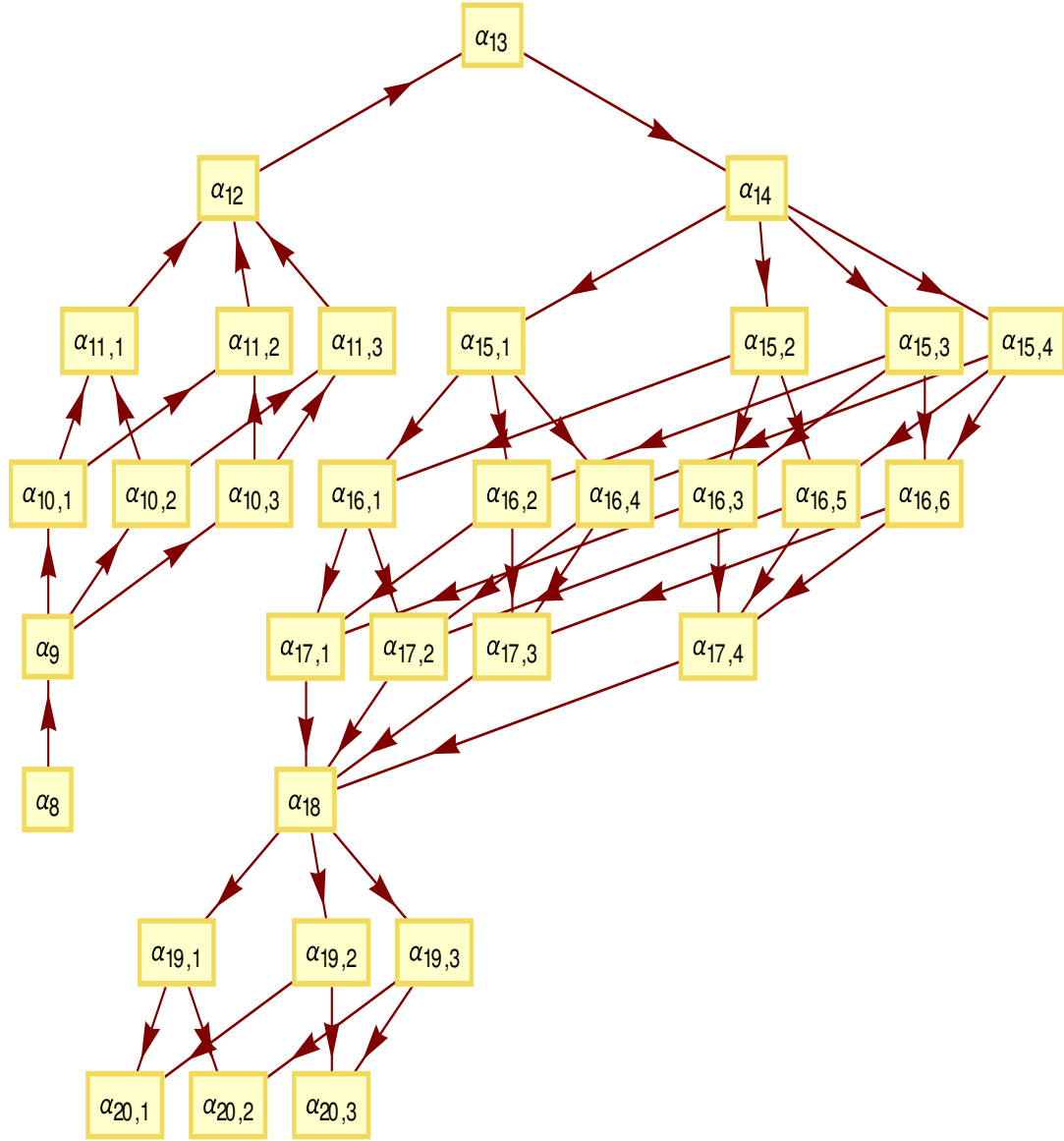
**Remark 3.11.** For the probability distribution supported by the nonhomogeneous Cantor distribution, considered in this paper, to obtain an optimal set of  $(n+1)$ -means one needs to know an optimal set of  $n$ -means. A closed formula is not known yet. Further investigation in this direction is still awaiting.

Using the induction formula given by Theorem 3.10, we obtain some results and observations about the optimal sets of  $n$ -means which are given in the following section.

#### 4. SOME RESULTS AND OBSERVATIONS

Let  $\alpha_n$  be an optimal set of  $n$ -means, i.e.,  $\alpha_n \in \mathcal{C}_n$ , and then for any  $a \in \alpha_n$ , we have  $a := a(\sigma) = S_\sigma(\frac{2}{3})$  for some  $\sigma := \sigma_1\sigma_2 \cdots \sigma_k \in \{1, 2\}^k$ ,  $k \geq 1$ . Moreover,  $a$  is the conditional expectation of the random variable  $X$  given that  $X$  is in  $J_\sigma$ , i.e.,  $a = S_\sigma(\frac{2}{3}) = E(X : X \in J_\sigma)$ . If  $\text{card}(\mathcal{C}_n) = k$  and  $\text{card}(\mathcal{C}_{n+1}) = m$ , then either  $1 \leq k \leq m$ , or  $1 \leq m \leq k$  (see Table 1). Moreover, by Theorem 3.10, we see that an optimal set at stage  $n$  can generate multiple distinct optimal sets at stage  $n+1$ , and multiple distinct optimal sets at stage  $n$  can produce one common optimal set at stage  $n+1$ ; for example from Table 1, we see that the number of  $\alpha_9 = 1$ , the number of  $\alpha_{10} = 3$ , the number of  $\alpha_{11} = 3$ , and the number of  $\alpha_{12} = 1$ . By  $\alpha_{n,i} \rightarrow \alpha_{n+1,j}$ , it is meant that the optimal set  $\alpha_{n+1,j}$  at stage  $n+1$  is produced from the optimal set  $\alpha_{n,i}$  at stage  $n$ , similar is the meaning for the notations  $\alpha_n \rightarrow \alpha_{n+1,j}$ , or  $\alpha_{n,i} \rightarrow \alpha_{n+1}$ , for example from Figure 1:

$$\begin{aligned} &\{\alpha_9 \rightarrow \alpha_{10,1}, \alpha_9 \rightarrow \alpha_{10,2}, \alpha_9 \rightarrow \alpha_{10,3}\}, \\ &\{\{\alpha_{10,1} \rightarrow \alpha_{11,1}, \alpha_{10,1} \rightarrow \alpha_{11,2}\}, \{\alpha_{10,2} \rightarrow \alpha_{11,1}, \alpha_{10,2} \rightarrow \alpha_{11,3}\}, \{\alpha_{10,3} \rightarrow \alpha_{11,2}, \alpha_{10,3} \rightarrow \alpha_{11,3}\}\}, \\ &\{\alpha_{11,1} \rightarrow \alpha_{12}, \alpha_{11,2} \rightarrow \alpha_{12}, \alpha_{11,3} \rightarrow \alpha_{12}\}. \end{aligned}$$

Figure 1: Tree diagram of the optimal sets from  $\alpha_8$  to  $\alpha_{20}$ .

$n$	$\text{card}(\mathcal{C}_n)$	$n$	$\text{card}(\mathcal{C}_n)$	$n$	$\text{card}(\mathcal{C}_n)$	$n$	$\text{card}(\mathcal{C}_n)$	$n$	$\text{card}(\mathcal{C}_n)$	$n$	$\text{card}(\mathcal{C}_n)$
5	1	18	1	31	15	44	120	57	7	70	6435
6	1	19	3	32	6	45	210	58	21	71	6435
7	2	20	3	33	1	46	252	59	35	72	5005
8	1	21	1	34	1	47	210	60	35	73	3003
9	1	22	1	35	1	48	120	61	21	74	1365
10	3	23	5	36	6	49	45	62	7	75	455
11	3	24	10	37	15	50	10	63	1	76	105
12	1	25	10	38	20	51	1	64	15	77	15
13	1	26	5	39	15	52	1	65	105	78	1
14	1	27	1	40	6	53	4	66	455	79	1
15	4	28	6	41	1	54	6	67	1365	80	10
16	6	29	15	42	10	55	4	68	3003	81	45
17	4	30	20	43	45	56	1	69	5005	82	120

TABLE 1. Number of  $\alpha_n$  in the range  $5 \leq n \leq 82$ .

Moreover, we see that

$$\alpha_9 = \{a(11), a(121), a(122), a(211), a(212), a(221), a(2221), a(22221), a(22222)\}$$

$$\text{with } V_9 = \frac{9805}{40108032} = 0.000244465;$$

$$\alpha_{10,1} = \{a(11), a(121), a(122), a(211), a(212), a(2211), a(2212), a(2221), a(22221), a(22222)\};$$

$$\alpha_{10,2} = \{a(11), a(121), a(122), a(211), a(221), a(2121), a(2122), a(2221), a(22221), a(22222)\};$$

$$\alpha_{10,3} = \{a(11), a(121), a(211), a(212), a(221), a(1221), a(1222), a(2221), a(22221), a(22222)\}$$

$$\text{with } V_{10} = \frac{7969}{40108032} = 0.000198688;$$

$$\alpha_{11,1} = \{a(11), a(121), a(122), a(211), a(2121), a(2122), a(2211), a(2212), a(2221), a(22221), a(22222)\};$$

$$\alpha_{11,2} = \{a(11), a(121), a(211), a(212), a(1221), a(1222), a(2211), a(2212), a(2221), a(22221), a(22222)\};$$

$$\alpha_{11,3} = \{a(11), a(121), a(211), a(221), a(1221), a(1222), a(2121), a(2122), a(2221), a(22221), a(22222)\}$$

$$\text{with } V_{11} = \frac{6133}{40108032} = 0.000152912;$$

$$\alpha_{12} = \{a(11), a(121), a(211), a(1221), a(1222), a(2121), a(2122), a(2211), a(2212), a(2221), a(22221), a(22222)\}$$

$$\text{with } V_{12} = \frac{4297}{40108032} = 0.000107136;$$

$$\alpha_{13} = \{a(111), a(112), a(121), a(211), a(1221), a(1222), a(2121), a(2122), a(2211), a(2212), a(2221), a(22221), a(22222)\}$$

$$\text{with } V_{13} = \frac{3481}{40108032} = 0.0000867906;$$

and so on.

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